

NON-COMMUTATIVE CREPANT RESOLUTIONS FOR SOME TORIC SINGULARITIES I

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ABSTRACT. We give a criterion for the existence of non-commutative crepant resolutions (NCCR's) for certain toric singularities. In particular we recover Broomhead's result that a 3-dimensional toric Gorenstein singularity has a NCCR. Our result also yields the existence of a NCCR for a 4-dimensional toric Gorenstein singularity which is known to have no toric NCCR.

1. INTRODUCTION

Throughout k is an algebraically closed field of characteristic zero. Let R be a normal Gorenstein domain. A *non-commutative crepant resolution* (NCCR) [DITV15, Leu12, ŠVdB17a, VdB04a, Wem] of R is an R -algebra of finite global dimension of the form $\Lambda = \text{End}_R(M)$ which in addition is Cohen-Macaulay as R -module and where M is a non-zero finitely generated reflexive R -module. In this note we discuss the existence of NCCR's for some toric singularities. The following is a combination of our main results.

Proposition 1.1. *Let G be an abelian reductive group over k (i.e. a product of a torus and a finite abelian group) and let W be a “generic” (see Definition 2.3 below) unimodular G -representation. Put $X := \text{Spec Sym}(W) = W^\vee$ and let $X^u := \{x \in X \mid 0 \in \overline{Gx}\}$ be the unstable locus. If $\dim X^u - \dim G \leq 1$ then $\text{Sym}(W)^G$ has a NCCR.*

Proof. This follows by combining Corollary 2.8, Proposition 3.1, and Remark 2.2 below. \square

Proposition 1.1 gives a relatively easy proof that three-dimensional toric Gorenstein singularities have a NCCR (see Corollary 3.2), a fact first proved by Broomhead [Bro12]. Actually Broomhead establishes the existence of a “toric” [Boc12] NCCR (M is a sum of reflexive ideals) which is much more difficult and relies on the theory of dimer models. In [ŠVdB17b] we give an alternative proof of Broomhead's result which is however still not easy.

In [ŠVdB17a, §10.1] we constructed toric NCCRs for toric rings coming from quasi-symmetric representations W (e.g. self-dual), and showed that in general toric NCCRs do not always exist. In other words, Broomhead's result does not

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extend to higher dimension. In fact in [ŠVdB17a, §10.1] we gave an example of a 4-dimensional toric Gorenstein singularity which does not have a toric NCCR. Using Proposition 1.1 we can now show that it nevertheless has a non-toric NCCR. See Example 3.3 below. On the other hand, Higashitani and Nakajima [HN17] recently constructed toric NCCRs for some natural examples of toric rings not coming from quasi-symmetric representations.

2. MAIN RESULT

All objects are defined over k . If \mathcal{X} is a stack then we write $D(\mathcal{X})$ for $D_{\text{Qch}}(\text{Mod}(\mathcal{O}_{\mathcal{X}}))$. If a reductive group G acts on an affine variety X and $\chi \in X(G)$ is a character of G then we write $X^{ss, \chi}$ for the open subset of X consisting of the χ -semi-stable points in X . In other words (following [Kin94]) $X^{ss, \chi}$ consists of the points $x \in X$ such that for every 1-parameter subgroups λ of G with the property that $\lim_{t \rightarrow 0} \lambda(t)x$ exists one has $\langle \lambda, \chi \rangle \geq 0$. We say that $x \in X$ is stable if it has closed orbit and finite stabilizer. We denote the locus of stable points in X by X^s . We have $X^s \subset X^{ss, \chi}$ for any χ . The next theorem extends [VdB04a, Thm. 5.1] to certain Deligne-Mumford stacks.

Theorem 2.1. *Let G be an abelian reductive group over k and let X be a smooth affine G -variety containing a G -stable point. Let $\chi \in X(G)$ be a character such that every point in $X^{ss, \chi}$ has finite stabilizer (i.e. $X^{ss, \chi}/G$ is a Deligne-Mumford stack) and assume in addition that $X^{ss, \chi} // G \rightarrow X // G$ has fibers of dimension ≤ 1 . Then $\text{coh}(X^{ss, \chi}/G)$ contains an object \mathcal{T} with the following properties.*

- (1) \mathcal{T} is a vector bundle on $X^{ss, \chi}$.
- (2) $\text{Ext}_{X^{ss, \chi}/G}^i(\mathcal{T}, \mathcal{T}) = 0$ for $i > 0$.
- (3) \mathcal{T} is a generator for $D(X^{ss, \chi}/G)$.

An object satisfying (1)(2)(3) is sometimes called a *tilting bundle*.

Remark 2.2. If G is an abelian reductive group acting linearly on an affine variety then $X^{ss, \chi}/G$ will be a Deligne-Mumford stack if $\chi \in X(G)$ is chosen generically. Indeed we may choose a closed embedding of X in a G -representation and for a representation the claim is a simple verification.

Whenever we are in the setting of Theorem 2.1 we will use the following maps between stacks whose definition should be clear

$$(1) \quad \begin{array}{ccc} X^{ss, \chi} & \xhookrightarrow{\tilde{\theta}} & X \\ \downarrow & & \downarrow \\ X^{ss, \chi}/G & \xhookrightarrow{\quad} & X/G \\ \downarrow \pi_s & & \downarrow \gamma_s \\ X^{ss, \chi} // G & \xrightarrow[\theta]{} & X // G \end{array} \quad \begin{array}{c} \pi \curvearrowright \\ \gamma \curvearrowright \end{array}$$

Under some genericity conditions one may obtain an NC(C)R from Theorem 2.1.

Definition 2.3. We say that a reductive group G acts *generically* on a smooth affine variety X if

- (1) X contains a point with closed orbit and trivial stabilizer.

(2) If $X^s \subset X$ is the locus of points that satisfy (1) then $\text{codim}(X - X^s, X) \geq 2$.

If W is a G -representation then we say that W is *generic* if G acts generically on $\text{Spec Sym}(W) \cong W^\vee$.

Corollary 2.4. *Let X, G, χ, \mathcal{T} be as in Theorem 2.1. Then $\mathcal{D}(X^{ss, \chi}/G) \cong \mathcal{D}(\Lambda)$ where $\Lambda = \text{End}_{X^{ss, \chi}/G}(\mathcal{T})$. One has $\text{gldim } \Lambda \leq \infty$. Moreover, if G acts generically on X then $\Lambda = \text{End}_R(T)$ where R is the coordinate ring of $X//G$ and $T = \Gamma(X^{ss, \chi}/G, \mathcal{T}) = \Gamma(X^{ss, \chi}, \mathcal{T})^G$ which is a reflexive R -module.*

Proof. The derived equivalence claim is clear. The derived equivalence implies $\text{gldim } \Lambda \leq \infty$ since $X^{ss, \chi}/G$ is smooth. We now refer to [SVdB17a, §3,4] for some generalities concerning reflexive sheaves we use below. Recall in particular that reflexive sheaves \mathcal{F}, \mathcal{G} on a normal variety Z form a rigid monoidal category with tensor product $\mathcal{F} \otimes \mathcal{G} := (\mathcal{F} \otimes_Z \mathcal{G})^{\vee\vee}$. Assume that $\text{codim}(X - X^s, X) \geq 2$. Then also $\text{codim}(X - X^{ss, \chi}, X) \geq 2$ and hence $\hat{\theta}_*$ defines a monoidal equivalence between the categories of reflexive sheaves on $X^{ss, \chi}$ and X . Using again the condition $\text{codim}(X - X^s, X) \geq 2$, taking G -invariants defines a monoidal equivalence between G -equivariant reflexive sheaves on X and reflexive sheaves on $X//G$. Since $\Gamma(X^{ss, \chi}, \mathcal{T})^G = \Gamma(X//G, (\hat{\theta}_* \mathcal{T})^G)$, $\Lambda = \Gamma(X//G, \hat{\theta}_*(\mathcal{T} \otimes_{X^{ss, \chi}} \mathcal{T}^\vee)^G)$ the conclusion follows. \square

Lemma 2.5. *Let X, G, χ be as in Theorem 2.1. Then θ is birational and $R\theta_* \mathcal{O}_{X^{ss, \chi}/G} = \mathcal{O}_{X//G}$. Finally $R^i \theta_* = 0$ for $i > 1$.*

Proof. Both $X^{ss, \chi}/G$ and $X//G$ contain X^s/G as an open subscheme. So they are birational. Both $X^{ss, \chi}/G$ and $X//G$ are quotients by reductive groups and hence they have rational singularities. This proves the claim about $R\theta_* \mathcal{O}_{X^{ss, \chi}/G}$. The last claim follows from the hypothesis that the fibers of θ have dimension ≤ 1 . \square

Lemma 2.6. *Let X, G, χ be as in Theorem 2.1. Assume in addition that $X = W^\vee$ where W is a generic unimodular G -representation. The map $f_s = \theta \pi_s$ is crepant and $\omega_{X^{ss, \chi}/G} \cong \mathcal{O}_{X^{ss, \chi}/G}$.*

Proof. The hypothesis imply that $\omega_{X//G}$ is invertible and moreover $\omega_{X//G} \cong \mathcal{O}_{X//G}$. A Deligne-Mumford stack is etale locally a quotient stack for a finite group and in particular $\omega_{X^{ss, \chi}/G}$ is a reflexive sheaf. We claim $f_s^* \omega_{X//G} = \omega_{X^{ss, \chi}/G}$ and hence in particular $\omega_{X^{ss, \chi}/G} \cong \mathcal{O}_{X^{ss, \chi}/G}$. This follows from the fact both $\omega_{X//G}$ and $\omega_{X^{ss, \chi}/G}$ are reflexive and f_s is the identity on $X^s/G \cong X^s//G$. \square

Remark 2.7. The assumption that W is a generic simplifies the proof of the previous lemma but it is in fact superfluous. This is a consequence of the theory of toric DM stacks [BH06].

Corollary 2.8. *Let X, G, χ be as in Theorem 2.1. Assume in addition that $X = W^\vee$ where W is a generic unimodular G -representation. Then $R = \text{Sym}(W)^G$ has a NCCR.*

Proof. Let $\mathcal{A} = \mathcal{E}nd_{X^{ss, \chi}/G}(\mathcal{T})$ where \mathcal{T} is as in Theorem 2.1. Then \mathcal{A} is a sheaf of algebras on $X^{ss, \chi}/G$. By Corollary 2.4 have to show that $Rf_{s,*} \mathcal{A}$ is Cohen-Macaulay. Using Lemma 2.6 we have by the same argument as [VdB04b, Lemma

3.2.9]

$$\begin{aligned}
\mathrm{RHom}_{X//G}(Rf_{s,*}\mathcal{A}, \omega_{X//G}) &= \mathrm{RHom}_{X^{ss,\chi}/G}(\mathcal{A}, f_s^! \omega_{X//G}) \\
&= \mathrm{RHom}_{X^{ss,\chi}/G}(\mathcal{A}, \omega_{X^{ss,\chi}/G}) \\
&= \mathrm{RHom}_{X^{ss,\chi}/G}(\mathcal{A}, \mathcal{O}_{X^{ss,\chi}/G}) \\
&= Rf_{s,*}\mathcal{A}^\vee \\
&= Rf_{s,*}\mathcal{A} \\
&= f_{s,*}\mathcal{A}
\end{aligned}$$

This finishes the proof. \square

For an open $U \subset X//G$ we write $\tilde{U} = U \times_{X//G} X \subset X$. We say that $D(X/G)$ is *locally generated* by a perfect object E if $D(\tilde{U}/G)$ is generated by $E|_{\tilde{U}}$ for every affine open $U \subset X//G$.

The following is a variant on [ŠVdB16, Lemma 3.5.4]. It can be deduced from the more general (see [OS03, Lemma 1.3, Theorem 5.7]). However it seems useful to give a direct proof in our simple setting.

Lemma 2.9. *Let G be a reductive group acting on an algebraic variety X such that a good quotient $\pi : X \rightarrow X//G$ exists and such that X/G is a Deligne-Mumford stack. Then $D(X/G)$ is locally generated by $V \otimes \mathcal{O}_X$ for a single finite dimensional representation V of G .*

Proof. We need to prove the existence of V such that $\pi_{s,*} \mathcal{H}om_{X/G}(V \otimes \mathcal{O}_X, \mathcal{F}) = 0$ implies $\mathcal{F} = 0$ for $\mathcal{F} \in \mathrm{Qch}(X/G)$ where $\pi_s : X/G \rightarrow X//G$ is the morphism of stacks associated to π .

If a certain V works then any bigger V works as well. Hence it follows that the existence of V is a local property for the étale topology on $X//G$. So we may assume that X is affine, and furthermore invoking the Luna slice theorem we may assume that π is of the form $G \times^H S \rightarrow (G \times^H S)//G \cong S//H$ where S is an étale slice at $x \in X$ with closed orbit and $H = \mathrm{Stab}(x)$. Since X/G is a Deligne-Mumford stack, H is finite. Let kH be the regular H -representation. Then $\mathcal{H}om_{S/H}(kH \otimes \mathcal{O}_S, \mathcal{F})$ implies $\mathcal{F} = 0$. Since $S/H \cong (G \times^H S)/G$, $kH \otimes_k \mathcal{O}_S$ corresponds to a G -equivariant vector bundle \mathcal{E} on $G \times^H S$. It now suffices to write \mathcal{E} as a quotient of $V \otimes \mathcal{O}_{G \times^H S}$ for some finite dimensional G -representation V . \square

Lemma 2.10. *Let G be a reductive group acting on an algebraic variety X which is projective over an affine variety and let \mathcal{M} be an ample G -equivariant line bundle on X . Let $X^{ss} \subset X$ be the semi-stable locus corresponding to the linearization given by \mathcal{M} and let $\pi : X^{ss} \rightarrow X^{ss}//G$ be the quotient map. Then up to replacing \mathcal{M} by a strictly positive multiple we may assume that $(\mathcal{M}|_{X^{ss}})^G$ is an ample line bundle on $X^{ss}//G$ generated by global sections such that moreover $\pi^*((\mathcal{M}|_{X^{ss}})^G) = \mathcal{M}$.*

Proof. Put

$$\Gamma_*(X) = \bigoplus_{n \geq 0} \Gamma(X, \mathcal{M}^{\otimes n})$$

Then $X^{ss}//G = \mathrm{Proj} \Gamma_*(X)^G$. There is an N such that the N 'th Veronese's of $\Gamma_*(X)$ and $\Gamma_*(X)^G$ are both generated in degree one. We then replace \mathcal{M} by $\mathcal{M}^{\otimes N}$. See [Bri, Proposition 1.35] for details. \square

Lemma 2.11. *Let G be a reductive group acting on an algebraic variety X which is projective over an affine variety and let \mathcal{M} be an ample G -equivariant line bundle on X . Let $X^{ss} \subset X$ be the semi-stable locus corresponding to the linearization given by \mathcal{M} and let $\pi : X^{ss} \rightarrow X^{ss} // G$, $\pi_s : X^{ss}/G \rightarrow X^{ss} // G$ be the associated quotient maps. We assume that X^{ss}/G is a Deligne-Mumford stack.*

In addition we assume that we have replaced \mathcal{M} by a strictly positive multiple such that $\mathcal{L} := (\mathcal{M} \mid X^{ss})^G \in \text{Pic}(X^{ss} // G)$ has the properties exhibited in Lemma 2.10. Put $Y = \text{Spec } \Gamma(X, \mathcal{O}_X)$. Let d be the maximum of the dimension of the fibers of $X^{ss} // G \rightarrow Y // G$. Let V be a finite dimensional representation of G such that $V \otimes \mathcal{O}_{X^{ss}}$ is a local generator for $D(X^{ss}/G)$ as in Lemma 2.9. Put $V = \bigoplus_{i=1}^n V_i$ with V_i irreducible and fix $m_i \in \mathbb{Z}$ for $i = 1, \dots, n$ and $l \geq 1$. Then

$$\bigoplus_{j=0}^d \bigoplus_{i=1}^n V_i \otimes \pi_s^*(\mathcal{L})^{\otimes lj+m_i} = \bigoplus_{j=0}^d \bigoplus_{i=1}^n V_i \otimes \mathcal{M}^{\otimes lj+m_i} \Big|_{X^{ss}}$$

is a compact generator for $D(X^{ss}/G)$.

Proof. Replacing \mathcal{M} by $\mathcal{M}^{\otimes l}$ we may assume $l = 1$. Put $\mathcal{E} = \bigoplus_{i=1}^n V_i \otimes \pi_s^*(\mathcal{L})^{\otimes m_i}$. Then since $\pi_s^*(\mathcal{L})$ is locally free on X^{ss}/G , \mathcal{E} is a local generator for $D(X^{ss}/G)$. We must prove that $\bigoplus_{j=0}^d \mathcal{E} \otimes \pi_s^*(\mathcal{L})^{\otimes j}$ is a generator for $D(X^{ss}/G)$.

Assume $\mathcal{F} \in D(X^{ss}/G)$ is such that $\text{RHom}_{X^{ss}/G}(\bigoplus_{j=0}^d \mathcal{E} \otimes \pi_s^*(\mathcal{L})^{\otimes j}, \mathcal{F}) = 0$. Then $\text{RHom}_{X^{ss} // G}(\bigoplus_{j=0}^d \mathcal{L}^{\otimes j}, \pi_{s,*} \text{RHom}_{X^{ss}/G}(\mathcal{E}, \mathcal{F})) = 0$. By [VdB04b, Lemma 3.2.2] this implies $\pi_{s,*} \text{RHom}_{X^{ss}/G}(\mathcal{E}, \mathcal{F}) = 0$. Since \mathcal{E} is a local generator this implies $\mathcal{F} = 0$. \square

Lemma 2.12. *Let G, X, χ be as in Theorem 2.1. Then there exist characters $(\chi_u)_{u=1, \dots, N}$ such that $\mathcal{L}_u = \chi_u \otimes \mathcal{O}_{X^{ss}, \chi}$ for $i = 1, \dots, N$ generate $D(X^{ss, \chi}/G)$ and such that moreover we have*

$$(2) \quad \text{Ext}_{X^{ss, \chi}/G}^i(\mathcal{L}_u, \mathcal{L}_u) = 0 \quad \text{for } i > 0$$

$$(3) \quad \text{Ext}_{X^{ss, \chi}/G}^i(\mathcal{L}_u, \mathcal{L}_v) = 0 \quad \text{for } i > 1$$

$$(4) \quad \text{Ext}_{X^{ss, \chi}/G}^1(\mathcal{L}_u, \mathcal{L}_v) = 0 \quad \text{for } u < v$$

Proof. According to Lemma 2.11 after replacing χ by some strict positive multiple $D(X^{ss, \chi}/G)$ has a compact generator of the form

$$\bigoplus_{j=0}^1 \bigoplus_{i=1}^n \mu_i \otimes \chi^{\otimes lj+m_i} \otimes \mathcal{O}_{X^{ss, \chi}}$$

and we put $\chi_1 = \mu_1 \otimes \chi^{m_1}$, $\chi_2 = \mu_1 \otimes \chi^{l+m_1}$, $\chi_3 = \mu_2 \otimes \chi^{m_2}$, \dots . Then (2) (3) follow directly from Lemma 2.5.

To make (4) true we choose $l, (m_i)_i$ in such a way that

$$m_1 \ll l + m_1 \ll m_2 \ll m_2 + l \ll m_3 \ll \dots$$

Then in (4) we have $\mathcal{L}_u = \mu_1 \otimes \chi^a \otimes \mathcal{O}_{X^{ss}}$, $\mathcal{L}_v = \mu_2 \otimes \chi^b \otimes \mathcal{O}_{X^{ss}}$ with $a \ll b$. Put $\mathcal{L} = \pi_{s,*}(\chi \otimes \mathcal{O}_{X^{ss}})$. By our choice of χ , \mathcal{L} is ample on $X^{ss} // G$ and $\pi_s^* \mathcal{L} = \chi \otimes \mathcal{O}_{X^{ss}}$. Using the projection formula we have

$$\begin{aligned} \text{RHom}_{X^{ss, \chi}/G}(\mathcal{L}_u, \mathcal{L}_v) &= R\Gamma(X^{ss, \chi} // G, \mathcal{L}^{\otimes b-a} \otimes \pi_{s,*}(\mu_2 \mu_1^{-1} \otimes \mathcal{O}_{X^{ss}})) \\ &= \Gamma(X^{ss, \chi} // G, \mathcal{L}^{\otimes b-a} \otimes \pi_{s,*}(\mu_2 \mu_1^{-1} \otimes \mathcal{O}_{X^{ss}})) \end{aligned}$$

where in the second line we use that \mathcal{L} is ample and $b - a \gg 0$. \square

Proof of Theorem 2.1. If E, F are objects in an abelian category \mathcal{A} such that $\text{Ext}_{\mathcal{A}}^1(E, F)$ (Yoneda Ext) is a finitely generated right $\mathcal{A}(E, E)$ -module with generators c_1, \dots, c_n then we define the corresponding semi-universal extension of E and F to be the middle term of the extension

$$0 \rightarrow F \rightarrow \bar{F} \rightarrow E^{\oplus n} \rightarrow 0$$

corresponding to $(c_i)_i$.

Let $(\mathcal{L}_u)_{u=1, \dots, N}$ be as in Lemma 2.12. Using the conditions (2,3,4) as in Lemma 2.12 we may construct the object \mathcal{T} by taking successive semi-universal extensions among the $(\mathcal{L}_u)_u$. See [HP14, Lemma 3.1] for details. In loc. cit. universal extensions are considered but the argument also works with semi-universal extensions. \square

3. COMBINATORIAL INTERPRETATION

If X is a representation then we write X^u for the G -unstable locus; i.e., $X^u = \{x \in X \mid 0 \in \overline{Gx}\}$. We let X, G, χ be as in Theorem 2.1, without a priori assuming that the fibers of $\theta : X^{ss, \chi} // G \rightarrow X // G$ have dimension ≤ 1 .

Proposition 3.1. *Assume $X = W^\vee$ for a G -representation W . Then if*

$$(5) \quad \dim X^u - \dim G \leq 1$$

the fibers of θ have dimension ≤ 1 .

Proof. We refer to the diagram (1). By semi-continuity is sufficient to bound the dimension of $\theta^{-1}(\bar{0})$. Now $\theta^{-1}(\bar{0}) = \pi(\gamma^{-1}(\bar{0}) \cap X^{ss, \chi})$. Since the fibers of π have constant dimension $\dim G$ we deduce

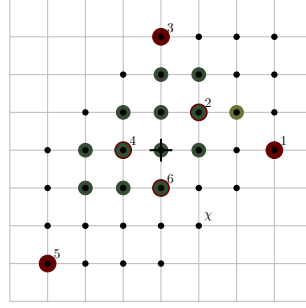
$$\begin{aligned} \dim \theta^{-1}(\bar{0}) &= \dim(\gamma^{-1}(\bar{0}) \cap X^{ss, \chi}) - \dim G \\ &= \dim(X^u \cap X^{ss, \chi}) - \dim G \leq \dim X^u - \dim G \quad \square \end{aligned}$$

Corollary 3.2. *Assume $X = W^\vee$ for a G -representation W . If W is generic and $\dim X // G = \dim X - \dim G \leq 3$ then the fibers of θ have dimension ≤ 1 .*

Proof. Let β_1, \dots, β_d be the weights of W . The fact that W is generic implies that for every $0 \neq \lambda \in Y(G)$ we have that there are at least two i such that $\langle \lambda, \beta_i \rangle > 0$. Hence $\dim X^u \leq \dim X - 2$. Thus $\dim X^u - \dim G \leq \dim X - \dim G - 2 \leq 3 - 2 = 1$. In other words (5) holds. \square

Note that (5) may hold for higher dimensional $X // G$.

Example 3.3. Consider the example [ŠVdB17a, §10.1]. Then we have that $G = G_m^2$ is a two dimensional torus and (after the identifying $X(G) \cong \mathbb{Z}^2$) the weights $(\beta_i)_i$ of W are given by $(3, 0), (1, 1), (0, 3), (-1, 0), (-3, -3), (0, -1)$ (see Figure 1). One checks $\dim X^u = 3$ and moreover W is generic and unimodular so that by Proposition 1.1 $R = \text{Sym}(W)^G \cong k[a, b, c, d, e]/(a^3b - cde)$ has a NCCR. However this NCCR is not toric which is the same as saying that it is not given by a module of covariants (a module of the form $M(U) = (U \otimes SW)^G$ for a finite dimensional G -representation U). In fact a NCCR given by a module of covariants does not exist in this case as is shown in loc. cit..


 FIGURE 1. \bullet^i weights, \bullet CM weights, \bullet \mathcal{L}

We have constructed an explicit NCCR for this example but we have not literally followed the proof of Proposition 1.1 which appeared computationally too expensive. Instead we obtain a NCCR using a similar but more adhoc procedure.

First we give some heuristic motivation for the construction. Assuming an appropriately strengthened version of the Bondal-Orlov conjecture asserting that all (stacky) commutative and non-commutative crepant resolutions are derived equivalent [BO02, IW13, VdB04a] the number of summands that we need for a non-commutative crepant resolution is given by the rank of K_0 of a (stacky) crepant commutative resolution of $\text{Spec } R$.

It is easy to verify that $\text{Spec } R$ as a (singular) toric variety corresponds to the fan given by the cone over a 3-dimensional polytope P shown in Figure 2. The volume of this polytope equals $13/6$, therefore the rank of K_0 of the stacky crepant resolution of $\text{Spec } R$, corresponding to a triangulation of P , is 13 (see Theorem A.1).

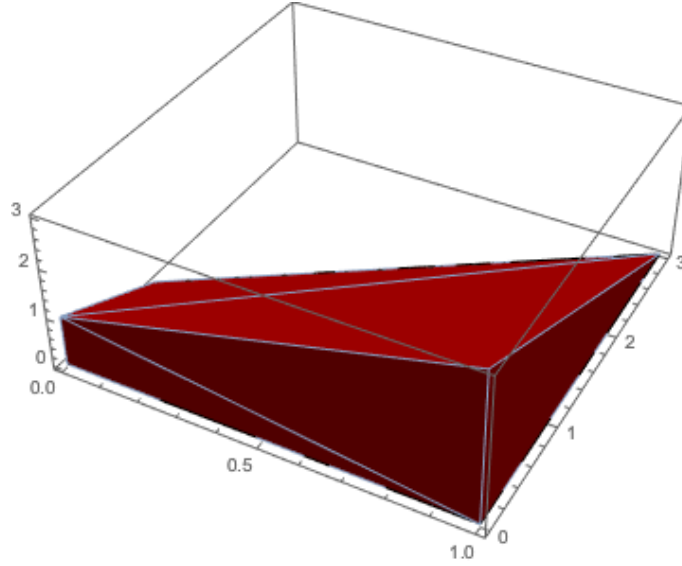


FIGURE 2

Let \mathcal{L} be given by weights corresponding to green dots in Figure 3.3 and let $\mathcal{L}' = \mathcal{L} \setminus \{(2, 1)\}$. The endomorphism ring $\text{End}_R(\bigoplus_{\mu \in \mathcal{L}'} M(\mu))$ is Cohen-Macaulay (see [ŠVdB17a, Example 10.1]). Since $|\mathcal{L}'| = 12$ we expect to need a single additional indecomposable R -module K such that $\Lambda = \text{End}_R(\bigoplus_{\mu \in \mathcal{L}'} M(\mu) \oplus K)$ is a NCCR. By loc. cit. K cannot be a module of covariants.

We define K by the exact sequence

$$(6) \quad 0 \rightarrow K \rightarrow M(0, -1) \oplus M(1, 1) \oplus M(-1, 1) \xrightarrow{\psi} M(2, 1) \rightarrow 0,$$

where $\psi(r_1, r_2, r_3) = r_1d + r_2a + r_3ab/c$. (Note that $M(0, -1) \cong (a, e)$, $M(1, 1) \cong (a, d)$, $M(-1, 1) \cong (a^2, ac, cd)$, $M(2, 1) \cong (a^2, ad, de)$.) Considering $M(\mu)$ as subsets of $\text{Sym}(W)$ we can write $\psi(r_1, r_2, r_3) = r_1x_2x_5 + r_2x_4 + r_3x_3x_5$.

It is easy to check that Λ is a Cohen-Macaulay R -module (using e.g. Macaulay2), suggesting that it might be a NCCR of R . Below we verify this fact by constructing an appropriate tilting bundle on a particular stacky resolution $X^{ss, \chi}/G$ of $\text{Spec } R$.

Let $\chi = (1, -2)$. We claim that $\mathcal{E} = \bigoplus_{\mu \in \mathcal{L}} \mu \otimes \mathcal{O}_{X^{ss, \chi}}$ generates $D(X^{ss, \chi}/G)$. One can use a similar algorithm as in the proof of [ŠVdB17a, Theorem 1.3.1]. (We refer to that proof also for some unexplained notation.) In loc. cit. the complexes $C_{\lambda, \mu}$ with cohomology supported on $X^{\lambda, \geq 0}$ relate projectives P_ν , $\nu \in X(T)$, in $D(X/G)$. Thus, if $\langle \chi, \lambda \rangle < 0$ then $C_{\lambda, \mu}$ is exact when restricted to $X^{ss, \chi}/G$ (recall that $X^{ss, \chi}$ consists of $x \in X$ such that if $\lambda \in Y(T)$ is such that $\lim_{t \rightarrow 0} \lambda(t)x$ exists then $\langle \lambda, \chi \rangle \geq 0$ which is equivalent to saying that $-\chi$ is in the cone generated by $(\beta_i)_{x_i \neq 0}$). Assume that $\tilde{\mathcal{L}} \subset X(T)$ is such that $\nu \otimes \mathcal{O}_{X^{ss, \chi}}$, $\nu \in \tilde{\mathcal{L}}$, belong to the subcategory of $D(X^{ss, \chi}/G)$ generated by \mathcal{E} (e.g. $\tilde{\mathcal{L}} = \mathcal{L}$). Then we may enlarge $\tilde{\mathcal{L}}$ by $\nu \in X(T)$ if for some $\langle \lambda, \chi \rangle < 0$ all components except for $\nu \otimes \mathcal{O}_{X^{ss, \chi}}$ of either of the complexes $C_{\lambda, \nu}$, $C_{\lambda, \nu - \sum_{\langle \lambda, \beta_i \rangle > 0} \beta_i}$ are of the form $\mu \otimes \mathcal{O}_{X^{ss, \chi}/G}$ for $\mu \in \tilde{\mathcal{L}}$. Note that if $\tilde{\mathcal{L}}$ contains $X(T) \cap \Sigma$, then we may enlarge $\tilde{\mathcal{L}}$ to $X(T)$. (See also the proof of [HLS16, Theorem 3.2].)

In our example we may easily verify by hand (or by computer, cf. [ŠVdB17a, Remark 11.3.2]) that we can indeed enlarge \mathcal{L} to $\Sigma \cap X(T)$ (where in this case $\Sigma \cap X(T)$ is given by weights corresponding to black dots in the above picture), and therefore \mathcal{E} generates $D(X^{ss, \chi}/G)$.

Since the endomorphism ring $\text{End}_R(\bigoplus_{\mu \in \mathcal{L}'} M(\mu))$ is Cohen-Macaulay, we have $\text{Ext}_{X^{ss, \chi}/G}^1(\mathcal{E}', \mathcal{E}') = 0$ for $\mathcal{E}' = \bigoplus_{\mu \in \mathcal{L}'} \mu \otimes \mathcal{O}_{X^{ss, \chi}}$ (see [VdB93, Corollary 3.3.2]).

Denote $M = \{(0, -1), (1, 1), (-1, 1)\} \subset \mathcal{L}'$. Let $\tilde{\psi} : \bigoplus_{\mu \in M} \mu \otimes \mathcal{O}_{X^{ss, \chi}} \rightarrow \mu_{(2, 1)} \otimes \mathcal{O}_{X^{ss, \chi}}$ be the lift of ψ to $X^{ss, \chi}/G$, and let \mathcal{K} be the lift of K (see the proof of Corollary 2.4). We claim that (6) induces an exact sequence

$$(7) \quad 0 \rightarrow \mathcal{K} \rightarrow \bigoplus_{\mu \in M} \mu \otimes \mathcal{O}_{X^{ss, \chi}} \xrightarrow{\tilde{\psi}} \mu_{(2, 1)} \otimes \mathcal{O}_{X^{ss, \chi}} \rightarrow 0.$$

Since $\tilde{\psi}$ is a restriction of the map $\Psi : \bigoplus_{\mu \in M} \mu \otimes \mathcal{O}_X \rightarrow \mu_{(2, 1)} \otimes \mathcal{O}_X$, induced from ψ , we need to check that the cokernel \mathcal{N} of this map has support in the complement of $X^{ss, \chi}$. The support of the cokernel is defined by the ideal (x_2x_5, x_4, x_3x_5) . Let $x = (x_1, \dots, x_6)$ belong to the support. Then either $x_2 = x_3 = x_4 = 0$ or $x_4 = x_5 = 0$. Since $-\chi$ does not lie in the cone generated by neither $\beta_1, \beta_5, \beta_6$ nor $\beta_1, \beta_2, \beta_3, \beta_6$, x does not belong to $X^{ss, \chi}$.

Moreover, any map from $\bigoplus_{\mu \in \mathcal{L}'} \mu \otimes \mathcal{O}_X$ to $\mu_{(2, 1)} \otimes \mathcal{O}_X$ factors through Ψ , since its image is zero in \mathcal{N} which easily follows from the fact that \mathcal{L}' does not intersect

the semigroups generated by $\beta_1, \beta_5, \beta_6$ and $\beta_1, \beta_2, \beta_3, \beta_6$, resp., shifted by $\mu_{2,1}$. Therefore, employing again the proof of Corollary 2.4, the map $\text{Hom}_{X^{ss,\chi}/G}(\mu_i \otimes \mathcal{O}_{X^{ss,\chi}}, \bigoplus_{\mu \in M} \mu \otimes \mathcal{O}_{X^{ss,\chi}}) \rightarrow \text{Hom}_{X^{ss,\chi}/G}(\mu_i \otimes \mathcal{O}_{X^{ss,\chi}}, \mu_{(2,1)} \otimes \mathcal{O}_{X^{ss,\chi}})$ induced from (7) is surjective. Thus, $\text{Ext}_{X^{ss,\chi}/G}^1(\mu_i \otimes \mathcal{O}_{X^{ss,\chi}}, \mathcal{K}) = 0$.

Applying $\text{Hom}_{X^{ss,\chi}/G}(-, \mu_i \otimes \mathcal{O}_{X^{ss,\chi}})$ and $\text{Hom}_{X^{ss,\chi}/G}(-, \mathcal{K})$ to (7) further implies that $\text{Ext}_{X^{ss,\chi}/G}^1(\mathcal{K}, \mu_i \otimes \mathcal{O}_{X^{ss,\chi}}) = 0$ and $\text{Ext}_{X^{ss,\chi}/G}^1(\mathcal{K}, \mathcal{K}) = 0$.

Since \mathcal{E} generates $D(X^{ss,\chi}/G)$, the same holds for $\mathcal{F} = \bigoplus_{\mu \in \mathcal{L}'} \mu \otimes \mathcal{O}_{X^{ss,\chi}} \oplus \mathcal{K}$ by (7), and we moreover have $\text{Ext}_{X^{ss,\chi}/G}^1(\mathcal{F}, \mathcal{F}) = 0$. Thus, $\text{End}_R(\bigoplus_{\mu \in \mathcal{L}'} M(\mu) \oplus K)$ is a NCCR of R by Corollary 2.8.

Remark 3.4. The discussion on the “universality” of Ψ in fact implies that ψ in (6) is the minimal $\text{add}(\bigoplus_{\mu \in \mathcal{L}'} M(\mu))$ -approximation of $M(2,1)$ in the sense that every map $M(\mu) \rightarrow M(2,1)$ for $\mu \in \mathcal{L}'$ factors through ψ .

Remark 3.5. Let $S = k[a, b, c, d, e]$. The module K introduced in the above example may also be described by a matrix factorization (d_0, d_1) of f :

$$d_0 = \begin{pmatrix} ab & 0 & ce & 0 \\ 0 & ab & -ac & -cd \\ -d & 0 & -a^2 & 0 \\ -a & -e & 0 & a^2 \end{pmatrix}, \quad d_1 = \begin{pmatrix} a^2 & 0 & ce & 0 \\ 0 & a^2 & -ac & cd \\ -d & 0 & -ab & 0 \\ a & e & 0 & ab \end{pmatrix},$$

where $d_0, d_1 : S^4 \rightarrow S^4$ and $K = \text{coker}(d_0)$.

APPENDIX A. GROTHENDIECK GROUP OF A TORIC DM STACK

Here we recall some results about the Grothendieck group of a toric DM stack. We mainly follow [BH06].

Let Σ be a fan, refining a cone over an $n-1$ -dimensional convex lattice polyhedron $P \times \{1\}$. Let $\mathbf{\Sigma}$ be a stacky fan $(\Sigma, (v_i)_{i=1}^l)$, where $v_i \in \mathbb{Z}^{n-1} \times \{1\}$ define 1-dimensional cones in Σ . We denote by $P_{\mathbf{\Sigma}}$ (resp. P_{Σ}) the corresponding toric DM stack (resp. toric variety). Note that $P_{\mathbf{\Sigma}}$ (resp. P_{Σ}) equals $Y^{ss,\chi}/G$ (resp. $Y^{ss,\chi}/\!/G$) for an action of $G \subset k^{*l}$ on $Y = k^n$ via characters determined by the images of e_i in $\mathbb{Z}^l/\rho(M) \cong X(G)$ ($\rho : m \mapsto (\langle m, v_i \rangle)_i$) and a generic $\chi \in X(G)$ (see [BH06, Section 2], [CLS11, Theorem 15.1.10]).

Let $\mu_i = \bar{e}_i \in X(G)$. We denote by R_i the class of the invertible sheaf $\mu_i \otimes \mathcal{O}_{Y^{ss,\chi}}$ in $K_0(P_{\mathbf{\Sigma}})$.

Theorem A.1. [BH06] *Let $P_{\mathbf{\Sigma}}$ be a toric DM stack. Let B the quotient of the Laurent polynomial ring $\mathbb{Z}[x_1, x_1^{-1}, \dots, x_l, x_l^{-1}]$ by the ideal generated by the relations*

- $\prod_{i=1}^l x_i^{\langle m, v_i \rangle} = 1$ for all $m \in M$,
- $\prod_{i \in I} (1 - x_i) = 0$ for any set $I \subseteq \{1, \dots, l\}$ such that $v_i, i \in I$, are not contained in any cone of Σ .

Then the map $\phi : B \rightarrow K_0(P_{\mathbf{\Sigma}})$ which sends x_i to R_i is an isomorphism. If P_{Σ} is a triangulation of a cone over a polyhedron P , then $\text{rk} K_0(P_{\mathbf{\Sigma}}) = (n-1)! \text{Vol}(P)$.

Proof. First part follows by [BH06, Theorem 4.10], while the last statement follows from [BH06, Remark 3.11, Theorem 5.3]. Indeed, we only need to show that $(1-t)^l \sum_{n \in N \cap \sigma} t^{\deg(n)}$ evaluated at 1 equals $(n-1)! \text{Vol}(P)$.

Note that $N \cap \sigma = \bigcup_{d \in \mathbb{N}} dP \times \{d\}$, and $\deg(n) = d$ for $n \in dP \times \{d\}$. Moreover, the number of lattice points in $dP \times \{d\}$ equals $\text{Ehr}_P(d)$, where Ehr denotes the

Ehrhart polynomial (see e.g. [CLS11, Theorem 9.4.2]). Since the degree of Ehr is $n - 1$ (as P is $n - 1$ -dimensional) and its leading coefficient equals $(n - 1)! \text{Vol}(P)$ (see e.g. [CLS11, Exercise 9.4.7]) we obtain that the above sum evaluated at 1 equals $(n - 1)! \text{Vol}(P)$. \square

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